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Aczél–Chebyshev Type Inequality for Positive Linear Functions[1]

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An Aczél–Chebyshev type inequality and some exact inequalities like the ones of Karamata for positive linear functionals are established. A counterexample on Chebyshev type inequality is given. © 2000 Academic Press

1. INTRODUCTION

Denote $\mathscr{R} = (-\infty, +\infty)$ and $\mathscr{R}^+ = [0, +\infty)$.

Let $\mathscr{E}$ be a nonempty set on $\mathscr{R}$ and let $\mathscr{L}$ be a class of nonnegative functions f: $\mathscr{E} \to \mathscr{R}^+$ and $\mathscr{F} \supset \mathscr{L}$ the class of functions f: $\mathscr{E} \to \mathscr{R}$ on which the functionals A are defined. We shall consider functionals A: $\mathscr{F} \to \mathscr{R}$ which satisfy the following conditions.

(1) $f \in \mathscr{L} \Rightarrow A(f) \geq 0$.

(2) $f \in \mathscr{F}$, $\lambda \in \mathscr{R} \Rightarrow \lambda f \in \mathscr{F}$, $A(\lambda f) = \lambda A(f)$.

(3) $1 \in \mathscr{F}$, $A(1) = 1$.

(4) $f, g \in \mathscr{F} \Rightarrow f + g \in \mathscr{F}$, $A(f + g) = A(f) + A(g)$.

If a functional A satisfies conditions (1)–(4), we say A is a *positive linear functional.*

Pečarić [5] obtained generalizations of fundamental mean inequality, Hölder's and Minkowski's, and their converse inequality. In the same paper, Pečarić also considered the functional $p(f) = (f_0^r - A(f^r))^{1/r}$ and gave some results which generalized the well-known Aczél, Popoviciu, and Bellman inequalities.

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In this paper, we shall consider the functional $p(f) = f_0 - A(f)$. In Section 2, we will establish an Aczél–Chebyshev type inequality for monotone functions $f(x)$ and $g(x)$ in the opposite sense. It is well known that if $f(x)$ and $g(x)$ are monotone in the same sense, we will have the reverse Chebyshev inequality (see [4]). Unfortunately, in this case, we cannot obtain the reverse Aczél–Chebyshev type inequality for $f(x)$ and $g(x)$ monotone in the same sense. In Section 3, we will give a counterexample which shows this so-called reverse inequality cannot hold. In Section 4, we will establish some exact inequalities like the ones of Karamata (see [2, 3]).

2. ACZÉL–CHEBYSHEV TYPE INEQUALITY

In order to prove our main result, we need the following lemma, which is a generalization of Chebyshev inequality for a positive linear functional.

LEMMA 2.1 (Chebyshev type inequality). *Let A be a positive linear functional with $A(1) = 1$. If $f(x)$ and $g(x)$ are monotone in the same sense, then*

$$A(fg) \geq A(f)A(g). \tag{2.1}$$

Moreover, if $f(x)$ and $g(x)$ are monotone in the opposite sense, then

$$A(fg) \leq A(f)A(g). \tag{2.2}$$

Proof. Assume f and g are increasing on $\mathscr{E}$. Then we have

$$\inf_{x\in\mathscr{E}} g(x) \leq A(g) \leq \sup_{x\in\mathscr{E}} g(x).$$

Therefore, there exists a $\xi \in \mathscr{E}$ satisfying

$$g(x) \geq A(g) \qquad \text{if } x \in \mathscr{E}, \quad x \geq \xi,$$

and

$$g(x) \leq A(g) \qquad \text{if } x \in \mathscr{E}, \quad x < \xi,$$

or satisfying

$$g(x) \geq A(g) \qquad \text{if } x \in \mathscr{E}, \quad x > \xi,$$

and

$$g(x) \leq A(g) \qquad \text{if } x \in \mathscr{E}, \quad x \leq \xi.$$

Hence, we have, for all $x \in \mathscr{E}$,

$$f(x)(g(x) - A(g)) \geq f(\xi)(g(x) - A(g)). \tag{2.3}$$

Acting by A on the two sides of Eqs. (2.3) yields

$$\begin{aligned} A(fg) - A(f)A(g) &= A(f(g - A(g))) \\ &\geq f(\xi)A(g - A(g)) = 0. \end{aligned}$$

For the other cases, the proof of Eq. (2.1) is similar. We omit the detail. ∎

THEOREM 2.1 (Aczél–Chebyshev type inequality). *Let A be a positive linear functional with $A(1) = 1$, and let f_0, g_0 be two nonnegative constants. If $f(x)$ and $g(x)$ are monotone in the opposite sense and $A(f) \geq 0$, $A(g) \geq 0$, $f_0 - A(f) \geq 0$, $g_0 - A(g) \geq 0$, then*

$$f_0 g_0 - A(fg) \geq (f_0 - A(f))(g_0 - A(g)). \tag{2.4}$$

Proof. If $f_0 = 0$ or $g_0 = 0$, then Eq. (2.4) follows from Eq. (2.2). Now assume $f_0 > 0$, and $g_0 > 0$. It is clear that Eq. (2.4) is equivalent to

$$1 - A(fg) \geq (1 - A(f))(1 - A(g)), \tag{2.5}$$

if we substitute f by f/f_0, g by g/g_0 in Eq. (2.4), respectively. And Eq. (2.5) is equivalent to the following inequality:

$$A(f)(1 - A(g)) + A(g) - A(fg) \geq 0. \tag{2.6}$$

Finally, Eq. (2.6) is true since

$$A(f)(1 - A(g)) \geq 0,$$

and, applying Eq. (2.2),

$$A(g) - A(fg) \geq A(g) - A(f)A(g) = A(g)(1 - A(f)) \geq 0.$$

∎

Remark. The conditions $A(f) \geq 0$ and $A(g) \geq 0$ in Theorem 2.1 are necessary. In fact, if we set

$$A(f) = \int_0^1 f(x)\,dx \tag{2.7}$$

and

$$f(x) = \begin{cases} -1, & x \in [0, \frac{1}{2}), \\ 0, & x \in [\frac{1}{2}, 1], \end{cases} \qquad g(x) = \begin{cases} \frac{1}{10}, & x \in [0, \frac{1}{2}), \\ \frac{1}{20}, & x \in [\frac{1}{2}, 1], \end{cases}$$

then $f(x)$ and $g(x)$ are monotone in the opposite sense and satisfy $A(f) < 0$ and $A(g) \geq 0$. But we have

$$1 - A(fg) = 1 + \tfrac{1}{20} < \left(1 + \tfrac{1}{2}\right)\left(1 - \tfrac{3}{40}\right) = (1 - A(f))(1 - A(g)).$$

3. A COUNTEREXAMPLE ON ACZÉL–CHEBYSHEV TYPE INEQUALITY

Now we give a counterexample which shows that if $f(x)$ and $g(x)$ are monotone in the same sense, generally, we cannot obtain so-called Aczél–Chebyshev type inequality.

Define the positive linear functional A as in Eq. (2.7). Obviously, the functional A satisfies condition $A(1) = 1$.

Let $0 \leq m \leq 10$ and

$$f(x) = g(x) = \begin{cases} m, & x \in \left[0, \frac{1}{10}\right], \\ 0, & x \in \left(\frac{1}{10}, 1\right]. \end{cases}$$

We have

$$\begin{aligned} (1 - A(f))(1 - A(g)) &= \left(1 - \int_0^1 f(x)\, dx\right)\left(1 - \int_0^1 g(x)\, dx\right) \\ &= \left(1 - \int_0^{1/10} m\, dx\right)^2 \\ &= \left(1 - \frac{m}{10}\right)^2 \end{aligned}$$

and

$$1 - A(fg) = 1 - \int_0^1 f(x)g(x)\, dx = 1 - \frac{m^2}{10}.$$

If $m = 3$, we have

$$\left(1 - \int_0^1 f(x)\, dx\right)\left(1 - \int_0^1 g(x)\, dx\right) > 1 - \int_0^1 f(x)g(x)\, dx.$$

If $m = \frac{1}{2}$, we obtain

$$\left(1 - \int_0^1 f(x)\, dx\right)\left(1 - \int_0^1 g(x)\, dx\right) < 1 - \int_0^1 f(x)g(x)\, dx.$$

This example shows that we cannot obtain the inequality

$$f_0 g_0 - A(fg) \leq (f_0 - A(f))(g_0 - A(g))$$

if $f(x)$ and $g(x)$ are monotone in the same sense. But we have the following

PROPOSITION 3.1. *Let A be a positive linear functional with $A(1) = 1$. If $0 \leq f(x) \leq f_0$ or $0 \leq g(x) \leq g_0$, we have*

$$(f_0 - A(f))(g_0 - A(g)) \leq f_0 g_0 - A(fg). \tag{3.1}$$

Proof. If $f_0 = 0$ or $g_0 = 0$, the inequality (3.1) is trivial. Now assume $f_0 > 0$, $g_0 > 0$. Then (3.1) is equivalent to the inequality

$$(1 - A(f))(1 - A(g)) \leq 1 - A(fg),$$

with $0 \leq f(x) \leq 1$ and $0 \leq g(x) \leq 1$. Then we have

$$1 - A(fg) \geq 1 - A(g) \geq (1 - A(f))(1 - A(g)).$$

∎

4. SOME EXACT INEQUALITIES

We say a functional A is *quasiconvex* (see [1]) if for $f, g \in \mathscr{F}$ and $0 < a < 1$, then

$$A(\alpha f + (1 - \alpha)g) \leq \max\{A(f), A(g)\}.$$

We say A is *quasiconcave* if

$$A(\alpha f + (1 - \alpha)g) \geq \min\{A(f), A(g)\}.$$

A functional $A(f)$ is said to be *continuous* for $f \in \mathscr{F}$ if, for arbitrary small $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|A(g) - A(f)| < \epsilon,$$

provided $g \in \mathscr{F}$ and

$$\|f - g\|_C < \delta.$$

If $A(f)$ is continuous for every $f \in \mathscr{S} \subset \mathscr{F}$, we say $A(f)$ is *continuous* on the set $\mathscr{S}$. Obviously, every positive linear functional is continuous.

Denote the characteristic function on the set Q by

$$X_Q(x) = \begin{cases} 1, & x \in Q, \\ 0, & x \notin Q. \end{cases}$$

We say $f(x)$ is a *step function* on $\mathscr{E}$ if

$$f(x) = \sum_{i=1}^{n} \lambda_i X_{Q_i}, \qquad \text{for some } n \in \mathscr{N},$$

where λ_i $(i = 1, \dots, n)$ are constants and

$$Q_i \subset \mathscr{E}, \qquad \bigcup_{i=1}^{n} Q_i = \mathscr{E} \qquad \text{and} \qquad Q_i \cap Q_j = \varnothing \qquad (i \neq j).$$

Let A be a positive linear functional with $A(1) = 1$. Define the functional

$$F(f, g) = \frac{f_0 g_0 - A(fg)}{(f_0 - A(f))(g_0 - A(g))},$$

for f and g satisfying $f_0 - A(f) > 0$ and $g_0 - A(g) > 0$.

Define the following classes of functions

$$B(m, M) = \{f(x) \mid 0 < m \le f(x) \le M, \forall x \in \mathscr{E}\},$$

$$B^*(m, M) = \left\{ \phi(x) \,\middle|\, \phi(x) = \sum_{i=1}^{n} \lambda_i X_{Q_i}(x) \text{ is a step function}, \; \lambda_i = m \text{ or } M \right\},$$

$$BI(m_1, M_1) = \{f(x) \mid m_1 \le f(x) \le M_1;\ f(x) \text{ is increasing on } \mathscr{E}\},$$

$$BD(m_2, M_2) = \{f(x) \mid m_2 \le f(x) \le M_2;\ f(x) \text{ is decreasing on } \mathscr{E}\}.$$

The purpose of this section is to calculate

$$\sup_{\substack{f \in B(m_1, M_1) \\ g \in B(m_2, M_2)}} F(f, g), \qquad \inf_{\substack{f \in B(m_1, M_1) \\ g \in B(m_2, M_2)}} F(f, g),$$

and

$$\sup_{\substack{f \in BI(m_1, M_1) \\ g \in BD(m_2, M_2)}} F(f, g), \qquad \inf_{\substack{f \in BI(m_1, M_1) \\ g \in BD(m_2, M_2)}} F(f, g).$$

In order to do that, we need the following lemmas.

LEMMA 4.1. *The functional $F(f, g)$ is quasiconvex and quasiconcave for f and g satisfying $f_0 - A(f) > 0$, $g_0 - A(g) > 0$.*

Proof. Assume $\alpha + \beta = 1$, $\alpha > 0$, $\beta > 0$. Since

$$F(\alpha f_1 + \beta f_2, g) = \frac{\alpha(f_0 - A(f_1))F(f_1, g) + \beta(f_0 - A(f_2))F(f_2, g)}{\alpha(f_0 - A(f_1)) + \beta(f_0 - A(f_2))},$$

then we have

$$\min\{F(f_1, g), F(f_2, g)\} \le F(\alpha f_1 + \beta f_2, g) \le \max\{F(f_1, g), F(f_2, g)\}.$$

Similarly, we can prove $F(f, g)$ is quasiconvex and quasiconcave for g. ∎

LEMMA 4.2. *Let A be a positive linear functional with $A(1) = 1$. If ϕ_1, ϕ_2 are step functions on $\mathscr{E}$,*

$$\phi_1(x) = \sum_{i=1,\, i\neq k}^{n} \lambda_i X_{Q_i}(x) + mX_{Q_k}(x),$$

$$\phi_2(x) = \sum_{i=1,\, i\neq k}^{n} \lambda_i X_{Q_i}(x) + MX_{Q_k}(x),$$

where $\lambda_i = m$ or M and $0 \le m < M \le 1$, then for $0 \le g(x) \le 1$,

$$\frac{1 - A(\phi_1 g)}{1 - A(\phi_1)} < \frac{1 - A(\phi_2 g)}{1 - A(\phi_2)}. \tag{4.1}$$

Proof. Obviously, (4.1) is the inequality

$$\frac{1 - A\left(g\sum_{i=1,\, i\neq k}^{n} \lambda_i X_{Q_i}\right) - mA\left(gX_{Q_k}\right)}{1 - A\left(\sum_{i=1,\, i\neq k}^{n} \lambda_i X_{Q_i}\right) - mA\left(X_{Q_k}\right)}$$
$$< \frac{1 - A\left(g\sum_{i=1,\, i\neq k}^{n} \lambda_i X_{Q_i}\right) - MA\left(gX_{Q_k}\right)}{1 - A\left(\sum_{i=1,\, i\neq k}^{n} \lambda_i X_{Q_i}\right) - MA\left(X_{Q_k}\right)}.$$

The above inequality is equivalent to

$$A\left(gX_{Q_k}\right)\left(1 - A\left(\sum_{i\neq k} \lambda_i X_{Q_i}\right)\right)$$
$$< A\left(X_{Q_k}\right)\left(1 - A\left(g\sum_{i\neq k} \lambda_i X_{Q_i}\right)\right).$$

It is clear that the above inequality is true. ∎

LEMMA 4.3. *Suppose $f_0 > M_1$ and $g_0 > M_2$. The functional $F(f, g)$ is continuous on the set $f \in B(m_1, M_1)$ and $g \in B(m_2, M_2)$, respectively.*

Proof. Since f and g are symmetric in the functional $F(f, g)$, it is sufficient to prove $F(f, g)$ is continuous on $f \in B(m_1, M_1)$. The continuity can be deduced easily from the equality

$$\begin{aligned} F(f, g) - F(f^*, g) &= \frac{1}{(f_0 - A(f))(f_0 - A(f^*))(g_0 - A(g))} \\ &\quad \times \{f_0 A((f^* - f)g) + f_0 g_0 A(f - f^*) \\ &\qquad + A(f^* - f) A(f^* g) + A(f^*) A((f - f^*)g)\}. \end{aligned}$$

∎

First we establish a general theorem.

THEOREM 4.1. *If functional $F(f)$ is continuous and quasiconvex on $\mathscr{F}$, then*

$$\sup_{f \in B(m, M)} F(f) = \sup_{\phi^* \in B^*(m, M)} F(\phi^*). \tag{4.2}$$

If functional $F(f)$ is continuous and quasiconcave on $\mathscr{F}$, then

$$\inf_{f \in B(m, M)} F(f) = \inf_{\phi^* \in B^*(m, M)} F(\phi^*). \tag{4.3}$$

Proof. Define step function on $\mathscr{E}$

$$\phi(x) = \sum_{i=1}^{n} \lambda_i X_{Q_i}(x),$$

where $m \le \lambda_i \le M$. If $m < \lambda_k < M$, then there exists an $\alpha \in (0, 1)$ such that

$$\lambda_k = \alpha m + (1 - \alpha) M.$$

Therefore,

$$\phi(x) = \alpha \phi_1(x) + (1 - \alpha) \phi_2(x),$$

where

$$\phi_1(x) = \sum_{i=1,\, i \ne k}^{n} \lambda_i X_{Q_i}(x) + m X_{Q_k}(x),$$

$$\phi_2(x) = \sum_{i=1,\, i \ne k}^{n} \lambda_i X_{Q_i}(x) + M X_{Q_k}(x).$$

Since $F(f)$ is quasiconvex, then

$$F(\phi) \leq \max\{F(\phi_1), F(\phi_2)\}.$$

Using the similar method for $i = 1, \dots, n$, we can obtain

$$F(\phi) \leq F(\phi^*),$$

where ϕ^* is a step function satisfying $\lambda_i = m$ or M $(i = 1, \dots, n)$. It is clear that for any measurable function $f(x)$ with $m \leq f(x) \leq M$ in Lebesgue sense and for arbitrary $\delta > 0$, there exists a step function $\phi(x)$ satisfying

$$\|f - \phi\|_C \leq \delta.$$

Then from the continuity of $F(f)$, we deduce

$$|F(f) - F(\phi)| < \epsilon.$$

Hence,

$$F(f) \leq F(\phi) + \epsilon \leq \epsilon + \sup_{\phi^* \in B^*(m, M)} F(\phi^*).$$

Since ϵ is arbitrarily small,

$$\sup_{f \in B(m, M)} F(f) \leq \sup_{\phi^* \in B^*(m, M)} F(\phi^*).$$

The inequality

$$\sup_{\phi^* \in B^*(m, M)} F(\phi^*) \leq \sup_{f \in B(m, M)} F(f)$$

is obvious. Therefore, Eq. (4.2) holds.

The proof of Eq. (4.3) is similar. ∎

THEOREM 4.2. *Suppose* $0 \leq m_1 < M_1 < f_0$ *and* $0 \leq m_2 < M_2 < g_0$. *The equalities*

$$\sup_{\substack{f \in BI(m_1, M_1) \\ g \in BD(m_2, M_2)}} F(f, g) = \sup_{\substack{f \in B(m_1, M_1) \\ g \in B(m_2, M_2)}} F(f, g) = \frac{f_0 g_0 - M_1 M_2}{(f_0 - M_1)(g_0 - M_2)} \tag{4.4}$$

and

$$\inf_{\substack{f \in BI(m_1, M_1) \\ g \in BD(m_2, M_2)}} F(f, g) = \inf_{\substack{f \in B(m_1, M_1) \\ g \in B(m_2, M_2)}} F(f, g) = \frac{f_0 g_0 - m_1 m_2}{(f_0 - m_1)(g_0 - m_2)} \tag{4.5}$$

hold.

Proof. Applying Lemmas 4.1 and 4.2 and Theorem 4.1 yields

$$\sup_{\substack{f \in BI(m_1, M_1) \\ g \in BD(m_2, M_2)}} F(f, g) \le \sup_{\substack{f \in B(m_1, M_1) \\ g \in B(m_2, M_2)}} F(f, g) = \sup_{\substack{f \in B^*(m_1, M_1) \\ g \in B^*(m_2, M_2)}} F(f, g). \tag{4.6}$$

Using Lemma 4.2, we get

$$\sup_{\substack{f \in B^*(m_1, M_1) \\ g \in B^*(m_2, M_2)}} F(f, g) \le F(\bar{f}, \bar{g}) = \frac{f_0 g_0 - M_1 M_2}{(f_0 - M_1)(g_0 - M_2)}, \tag{4.7}$$

where $\bar{f}(x) \equiv M_1$, $\bar{g}(x) = M_2$ $(\forall x \in \mathscr{E})$. Since $\bar{f} \in B^*(m_1, M_1)$ and $\bar{g} \in B^*(m_2, M_2)$, Eq. (4.4) follows from Eqs. (4.6) and (4.7).

As to Eq. (4.5), the proof is similar. ∎

COROLLARY 4.2.1. *Let A be a positive linear functional with $A(1) = 1$. Suppose $f \in B(m_1, M_1)$ and $g \in B(m_2, M_2)$, where $0 \le m_1 < M_1 < f_0$ and $0 \le m_2 < M_2 < g_0$. Then the inequality*

$$\begin{aligned} &K_*(f_0 - A(f))(g_0 - A(g)) \\ &\qquad \le f_0 g_0 - A(fg) \le K^*(f_0 - A(f))(g_0 - A(g)), \end{aligned}$$

holds, *where*

$$K_* = \frac{f_0 g_0 - m_1 m_2}{(f_0 - m_1)(g_0 - m_2)}, \qquad K^* = \frac{f_0 g_0 - M_1 M_2}{(f_0 - M_1)(g_0 - M_2)}.$$

Denote

$$F(f_1, f_2, \dots, f_n) = \frac{\prod_{k=1}^n \left(G_k - A\left(\prod_{k=1}^n f_k\right)\right)}{\prod_{k=1}^n (G_k - A(f_k))},$$

where G_k $(k = 1, \dots, n)$ are constants and $G_k - A(\prod_{k=1}^n f_k) > 0$ $(k = 1, \dots, n)$.

COROLLARY 4.2.2. *Suppose* $f_k \in B(m_k, M_k)$ *and* $0 \le m_k < M_k < G_k$ $(k = 1, \ldots, n)$. *Then we have*

$$\sup_{\substack{f_k \in B(m_k, M_k)\\ k=1,\ldots,n}} F(f_1, \ldots, f_n) = \frac{\prod_{k=1}^n G_k - \prod_{k=1}^n M_k}{\prod_{k=1}^n (G_k - M_k)},$$

and

$$\inf_{\substack{f_k \in B(m_k, M_k)\\ k=1,\ldots,n}} F(f_1, \ldots, f_n) = \frac{\prod_{k=1}^n G_k - \prod_{k=1}^n m_k}{\prod_{k=1}^n (G_k - m_k)}.$$

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